Revisiting Postulates for Inconsistency Measures

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Résumé

Cet article est un examen critique de postulats pour les mesures d'inconsistance. Il présente des objections envers deux postulats bien connus. Il contient aussi des observations plus générales et introduit finalement un nouvel ensemble de postulats.

Abstract

Postulates for inconsistency measures are examined, the set of postulates due to Hunter and Konieczny being the starting point. Objections are raised against a few individual postulates. More general shortcomings are discussed and a new series of postulates is introduced.

1 Introduction

In a couple of influential papers [11] [12], Hunter and Konieczny have introduced postulates for inconsistency measures over knowledge bases. Let us first make it clear that the phrase "inconsistency measure" refers to the informal meaning of a measure, not to the usual formal definition whose countable additivity requirement would actually leave no choice for an inconsistency measure, making all minimal inconsistent knowledge bases in each cardinality to count as equally inconsistent (unless making some *consistent* formulas to count as more *inconsistent* than others !). However, we stick with the usual range $R^+ \cup \{\infty\}$ (so that the codomain is totally ordered and 0 is the least element). Here is the intuition : the higher the amount of inconsistency in the knowledge base, the greater the number returned by the inconsistency measure.

Let us emphasize that we deal with postulates for inconsistency measures that account for a raw amount of inconsistency : E.g., it will clearly appear below that an inconsistency measure I satisfying the (Monotony) postulate due to Hunter-Konieczny precludes I to be a ratio (except for quite special cases such as pointed out in [12]).

Out of the properties listed by [12] [22], we will ignore both super-additivity :
• if $K \cap K' = \emptyset$ then $I(K \cup K') \geq I(K) + I(K')$
and MI-separability :
• $I(K \cup K') = I(K) + I(K')$ if $\{MI(K), MI(K')\}$ is a partition of $MI(K \cup K')$
($MI(X)$ is the set of minimal inconsistent subsets of X) that are significant but over-demanding as postulates (the latter is based on minimal inconsistent subsets, an approach which we argue against at length).

2 HK Postulates

Hunter and Konieczny consider a propositional language[1] $\mathcal{L}$ for classical logic $\vdash$. Finite sequences over $\mathcal{L}$ are called belief bases. $\mathcal{K}_{\mathcal{L}}$ is comprised of all belief bases over $\mathcal{L}$, in set-theoretic form (i.e., a member of $\mathcal{K}_{\mathcal{L}}$ is an ordinary set[2]).

According to Hunter and Konieczny, a function I over belief bases is an inconsistency measure if it satisfies the following properties, $\forall K, K' \in \mathcal{K}_{\mathcal{L}}$, $\forall \alpha, \beta \in \mathcal{L}$

- $I(K) = 0$ iff $K \nvdash \bot$ (Consistency Null)
- $I(K \cup K') \geq I(K)$ (Monotony)
- If α is free[3] for K then $I(K \cup \{\alpha\}) = I(K)$ (Free Formula Independence)
- If $\alpha \vdash \beta$ and $\alpha \nvdash \bot$ then $I(K \cup \{\alpha\}) \geq I(K \cup \{\beta\})$ (Dominance)

Letting $\equiv^s$ denote pointwise equivalence, another property from [22] is
- If $K \equiv^s K'$ then $I(K) = I(K')$ (Irrelevance of Syntax)

It implies $I(\{\alpha\}) = I(\{\beta\})$ for $\alpha \vdash \bot$ and $\beta \vdash \bot$, too

[1] For simplicity, we consider the language generated from the complete set of connectives $\{\neg, \wedge, \vee\}$.

[2] In the conclusion, we mention the case of multisets.

[3] A formula φ is free for X iff $Y \cup \{\alpha\} \vdash \bot$ for no consistent subset Y of X.

restrictive for a postulate. As to its consistent version, it is dealt with in Section 4 where it is called (Swap).

We argue against (Free Formula Independence) and (Dominance) in Section 3. We list in Section 4 some consequences of HK postulates, stressing the need for more general principles in each case. Section 5 is devoted to a major issue, replacement of equivalent subsets. Throughout Section 6, we introduce various postulates supplementing the original ones, ending with a new axiomatization. Section 7 can be viewed as a kind of rejoinder backing both (Free Formula Independence) and (Monotony) through the main new postulate.

3 Objections to HK Postulates

3.1 Objection to (Dominance)

In contrapositive form, (Dominance) says :

$$\text{For } \alpha \vdash \beta, \text{ if } I(K \cup \{\alpha\}) < I(K \cup \{\beta\}) \text{ then } \alpha \vdash \bot \qquad (1)$$

although it makes sense that the left hand side holds without $\alpha \vdash \bot$. An example is as follows. Let $K = \{a \wedge b \wedge c \wedge \cdots \wedge z\}$. Take $\beta = \neg a \vee (\neg b \wedge \neg c \wedge \cdots \wedge \neg z)$ while $\alpha = \neg a$. We may hold $I(K \cup \{\alpha\}) < I(K \cup \{\beta\})$ on the following grounds :

- The inconsistency in $I(K \cup \{\alpha\})$ is $\neg a$ vs a.
- The inconsistency in $I(K \cup \{\beta\})$ is either as above (i.e., $\neg a$ vs a) or it is $\neg b \wedge \neg c \wedge \cdots \wedge \neg z$ vs $b \wedge c \wedge \cdots \wedge z$ that may be viewed as more inconsistent than the case $\neg a$ vs a, hence, $\{a \wedge b \wedge c \wedge \cdots \wedge z\} \cup \{\neg a \vee (\neg b \wedge \neg c \wedge \cdots \wedge \neg z\}$ can be taken as more inconsistent overall than $\{a \wedge b \wedge c \wedge \cdots \wedge z\} \cup \{\neg a\}$ thereby violating (1) because $\alpha \nvdash \bot$ here.

3.2 Objection to (Free Formula Independence)

Unfolding the definition of a free formula, (Free Formula Independence) is :

$$\text{If } K' \cup \{\alpha\} \vdash \bot \text{ for no consistent subset } K' \text{ of } K \text{ then } I(K \cup \{\alpha\}) = I(K) \qquad (2)$$

The following is a case illustrating why (Free Formula Independence) is dubious : $K = \{a \wedge \neg a \wedge \neg b, \neg a \wedge b \wedge \neg b\}$. Take $\alpha = a \wedge b$. Clearly, $a \wedge b$ is a free formula of $K \cup \{a \wedge b\}$ but its rightmost conjunct causes a contradiction with a *consistent* part of a formula of K and similarly does its leftmost conjunct, hence $K \cup \{a \wedge b\}$ can be viewed as more inconsistent than K, resulting in a violation for (2).

A similar example is given in [12] but ours can be turned into a case in which no minimal inconsistent subset is a singleton set (consider $K = \{a \wedge c, b \wedge \neg c\}$ and $\alpha = \neg a \vee \neg b$).

4 Consequences of HK Postulates

Proposition 1 *(Monotony) entails*

- *if* $I(K \cup \{\alpha \wedge \beta\}) = I(K \cup \{\alpha, \beta\})$ *then* $I(K \cup \{\alpha \wedge \beta\}) \geq I(K \cup \{\beta\})$

Proof Assume $I(K \cup \{\alpha \wedge \beta\}) = I(K \cup \{\alpha, \beta\})$. According to (Monotony), $I(K \cup \{\alpha, \beta\}) \geq I(K \cup \{\beta\}$. Hence the result. ∎

So, if I conforms with adjunction (roughly speaking, it means identifying $\{\alpha, \beta\}$ with $\{\alpha \wedge \beta\}$) then I respects the idea that adding a conjunct cannot make the amount of inconsistency to decrease.

Notation. $\alpha \equiv \beta$ denotes that both $\alpha \vdash \beta$ and $\beta \vdash \alpha$ hold. Also, $\alpha \equiv \beta \vdash \gamma$ is an abbreviation for $\alpha \equiv \beta$ and $\beta \vdash \gamma$ (so, $\alpha \equiv \beta \nvdash \gamma$ means that $\alpha \equiv \beta$ and $\beta \nvdash \gamma$).

Proposition 2 *(Free Formula Independence) entails*

- *if* $\alpha \equiv \top$ *then* $I(K \cup \{\alpha\}) = I(K)$ (Tautology Independence)

Proof A tautology is trivially a free formula for any K. ∎

Unless $\beta \nvdash \bot$, there is however no guarantee that the following holds :

- if $\alpha \equiv \top$ then $I(K \cup \{\alpha \wedge \beta\}) = I(K \cup \{\beta\})$ ($\top$-conjunct Independence)

Proposition 3 *(Dominance) entails*

- $I(K \cup \{\alpha_1, \ldots, \alpha_n\}) = I(K \cup \{\beta_1, \ldots, \beta_n\})$ *if* $\alpha_i \equiv \beta_i \nvdash \bot$ *for* $i = 1..n$ (Swap)

Proof For $i = 1..n$, $\alpha_i \equiv \beta_i$ and (Dominance) can be applied in both directions. $I(K \cup \{\beta_1, \ldots, \beta_{i-1}, \alpha_i, \ldots, \alpha_n\}) = I(K \cup \{\beta_1, \ldots, \beta_i, \alpha_{i+1}, \ldots, \alpha_n\})$ for $i = 1..n$. ∎

Proposition 3 fails to guarantee that I be independent of any consistent subset of the knowledge base being replaced by an equivalent (consistent) set of formulas :

- if $K' \nvdash \bot$ and $K' \equiv K''$ then $I(K \cup K') = I(K \cup K'')$ (Exchange)

Proposition 3 at least guarantees that any consistent formula of the knowledge base can be replaced by an equivalent formula without altering the result of the inconsistency measure. Of course, postulates for inconsistency measures are expected *not* to entail $I(K \cup \{\alpha\}) = I(K \cup \{\beta\})$ for $\alpha \equiv \beta$ such that $\alpha \vdash \bot$. However, some subcases are desirable such as $I(K \cup \{\alpha \vee \alpha\}) = I(K \cup \{\alpha\})$, $I(K \cup \{\alpha \wedge \beta\}) = I(K \cup \{\beta \wedge \alpha\})$, and so on, in full generality (even for $\alpha \vdash \bot$) but Proposition 3 fails to ensure any of these.

Proposition 4 *(Dominance) entails*

- *if* $\alpha \wedge \beta \nvdash \perp$ *then* $I(K \cup \{\alpha \wedge \beta\}) \geq I(K \cup \{\beta\})$

Proof Applying (Dominance) to the valid entailment $\alpha \wedge \beta \vdash \beta$ yields the result. ■

Proposition 4 means that I respects the idea that adding a conjunct cannot make the amount of inconsistency to decrease, in the case of a consistent conjunction (however, one really wonders why this not guaranteed to hold in more cases ?).

Proposition 5 *Due to (Dominance) and (Monotony)*

- *For* $\alpha \in K$*, if* $\alpha \nvdash \perp$ *and* $\alpha \vdash \beta$ *then* $I(K \cup \{\beta\}) = I(K)$

Proof $I(K \cup \{\alpha\}) = I(K)$ as $\alpha \in K$. By (Dominance), $I(K \cup \{\alpha\}) \geq I(K \cup \{\beta\})$. Therefore, $I(K) \geq I(K \cup \{\beta\})$. The converse holds due to (Monotony). ■

Proposition 5 guarantees that a consequence of a consistent formula of the knowledge base can be added without altering the result of the inconsistency measure. What about a consequence of a consistent subset of the knowledge base ? Indeed, Proposition 5 is a special case of

(A_n) For $\{\alpha_1, \ldots, \alpha_n\} \subseteq K$, if $\{\alpha_1, \ldots, \alpha_n\} \nvdash \perp$ and $\{\alpha_1, \ldots, \alpha_n\} \vdash \beta$ then $I(K \cup \{\beta\}) = I(K)$

That is, Proposition 5 guarantees (A_n) only for $n = 1$ but what is the rationale for stopping there ?

Example 1 *Let* $K = \{\neg b, a \wedge b, b \wedge c\}$*. Proposition 5 ensures that* $I(K \cup \{a, c\}) = I(K \cup \{a\}) = I(K \cup \{c\}) = I(K)$*. Although* $a \wedge c$ *behaves as* a *and* c *with respect to all contradictions in* K *(i.e.,* $a \wedge b$ *vs* $\neg b$ *and* $b \wedge c$ *vs* $\neg b$*), HK postulates fail to ensure* $I(K \cup \{a \wedge c\}) = I(K)$*, no matter how natural the equality is.*

5 Two Postulates for Replacement of Equivalent Subsets

5.1 Replacing consistent equivalent subsets : The value of (Exchange)

To start with, (Exchange) is not a consequence of (Dominance) and (Monotony). An example is $K_1 = \{a \wedge c \wedge e, b \wedge d \wedge \neg e\}$ and $K_2 = \{a \wedge e, c \wedge e, b \wedge d \wedge \neg e\}$. By (Exchange), $I(K_1) = I(K_2)$ but HK postulates do not impose the equality. Next are a few results showing properties of (Exchange).

Proposition 6 *The following items are pairwise equivalent :*

- *(Exchange)*
- *The family* $(A_n)_{n \geq 1}$
- *If* $K' \nvdash \perp$ *and* $K' \equiv K''$ *then* $I(K \cup K') = I((K \setminus K') \cup K'')$
- *If* $K' \nvdash \perp$ *and* $K \cap K' = \emptyset$ *and* $K' \equiv K''$ *then* $I(K \cup K') = I(K \cup K'')$
- *If* $\{K_1, \ldots, K_n\}$ *is a partition of* $K \setminus K_0$ *where* $K_0 = \{\alpha \in K \mid \alpha \vdash \perp\}$ *such that* $K_i \nvdash \perp$ *and* $K'_i \equiv K_i$ *for* $i = 1..n$ *then* $I(K) = I(K_0 \cup K'_1 \cup \cdots \cup K'_n)$

Proof Assume (A_n) for all $n \geq 1$ and $K' \equiv K'' \nvdash \perp$. (i) Let $K' = \{\alpha_1, \ldots, \alpha_m\}$. Define $\langle K'_j \rangle_{j \geq 0}$ where $K'_0 = K \cup K''$ and $K'_{j+1} = K'_j \cup \{\alpha_{j+1}\}$. It is clear that $K'' \nvdash \perp$ and $K'' \vdash \alpha_{j+1}$ and $K'' \subseteq K'_j$. Hence, (A_n) can be applied to K'_j and this gives $I(K'_j) = I(K'_j \cup \{\alpha_{j+1}\}) = I(K'_{j+1})$. Overall, $I(K'_0) = I(K'_m)$. I.e., $I(K \cup K'') = I(K \cup K' \cup K'')$. (ii) Let $K'' = \{\beta_1, \ldots, \beta_p\}$. Consider the sequence $\langle K''_j \rangle_{j \geq 0}$ where $K''_0 = K \cup K'$ and $K''_{j+1} = K''_j \cup \{\beta_{j+1}\}$. Clearly, $K' \nvdash \perp$ and $K' \vdash \beta_{j+1}$ and $K' \subseteq K''_j$. Hence, (A_n) can be applied to K''_j and this gives $I(K''_j) = I(K''_j \cup \{\beta_{j+1}\}) = I(K''_{j+1})$. Overall, $I(K''_0) = I(K''_p)$. I.e., $I(K \cup K') = I(K \cup K' \cup K'')$. Combining the equalities, $I(K \cup K') = I(K \cup K'')$. That is, the family $(A_n)_{n \geq 1}$ entails (Exchange).
We now show that the family $(A_n)_{n \geq 1}$ is entailed by the third item in the statement of Proposition 6, denoted (Exchange′), which is :

$$\text{If } K' \nvdash \perp \text{ and } K' \equiv K'' \text{ then}$$
$$I(K \cup K') = I((K \setminus K') \cup K'').$$

Let $\{\alpha_1, \ldots, \alpha_n\} \subseteq K$ such that $\{\alpha_1, \ldots, \alpha_n\} \nvdash \perp$ and $\{\alpha_1, \ldots, \alpha_n\} \vdash \beta$. So, $\{\alpha_1, \ldots, \alpha_n\} \equiv \{\alpha_1, \ldots, \alpha_n, \beta\}$. For $K' = \{\alpha_1, \ldots, \alpha_n\}$, $K'' = \{\alpha_1, \ldots, \alpha_n, \beta\}$ (Exchange) gives $I(K) = I((K \setminus \{\alpha_1, \ldots, \alpha_n\}) \cup \{\alpha_1, \ldots, \alpha_n, \beta\} = I(K \cup \{\beta\})$.
By transitivity, we have thus shown that (Exchange) is entailed by (Exchange′). Since the converse is obvious, the equivalence between (Exchange), (Exchange′) and the family $(A_n)_{n \geq 1}$ holds.

It is clear that the fourth item in the statement of Proposition 6 is equivalent with (Exchange).

Consider now (Exchange″), the last item in the statement of Proposition 6 :

If $\{K_1, \ldots, K_n\}$ is a partition of $K \setminus K_0$ such that $K_0 = \{\alpha \in K \mid \alpha \vdash \perp\}$ and $K'_i \equiv K_i \nvdash \perp$ for $i = 1..n$ then $I(K) = I(K_0 \cup K'_1 \cup \cdots \cup K'_n)$.

(i) Assume (Exchange′). We now prove (Exchange″). Let $\{K_1, \ldots, K_n\}$ be a partition of $K \setminus K_0$ satisfying the conditions of (Exchange″). Trivially, $I(K) = I(K_0 \cup K \setminus K_0) = I(K_0 \cup K_1 \cup \cdots \cup K_n)$. Then, $K_i \setminus K_n = K_i$ for $i = 1..n-1$. Applying (Exchange′) yields $I(K_0 \cup K_1 \cup \cdots \cup K_n) = I(K_0 \cup K_1 \cup \cdots \cup K'_n)$

hence $I(K) = I(K_0 \cup K_1 \cup \cdots \cup K'_n)$. Applying (Exchange′) iteratively upon K_{n-1}, K_{n-2}, …, K_1 gives $I(K) = I(K_0 \cup K'_1 \cup \cdots \cup K'_n)$.
(ii) Assume (Exchange″). We now prove (Exchange′). Let $K' \nvdash \bot$ and $K'' \equiv K'$. Clearly, $(K \cup K')_0 = K_0$ and $(K \cup K') \setminus (K \cup K')_0 = (K \setminus K_0) \cup K'$. As each formula in $K \setminus K_0$ is consistent, $K \setminus K_0$ can be partitioned into $\{K_1, \ldots, K_n\}$ such that $K_i \nvdash \bot$ for $i = 1..n$ (take $n = 0$ in the case that $K = K_0$). Then, $\{K_1 \setminus K', \ldots, K_n \setminus K', K'\}$ is a partition of $(K \setminus K_0) \cup K'$ satisfying the conditions in (Exchange″). Now, $I(K \cup K') = I(K_0 \cup (K_1 \setminus K') \cup \cdots \cup (K_n \setminus K') \cup K')$. Applying (Exchange″) with each K_i substituting itself and K'' substituting K', we obtain $I(K \cup K') = I(K_0 \cup (K_1 \setminus K') \cup \cdots \cup (K_n \setminus K') \cup K'')$. That is, $I(K \cup K') = I((K \setminus K') \cup K'')$. ■

Proposition 7 *(Exchange) entails (Swap).*

Proof Taking advantage of transitivity of equality, it will be sufficient to prove $I(K \cup \{\beta_1, \ldots, \beta_{i-1}, \alpha_i, \ldots, \alpha_n\}) = I(K \cup \{\beta_1, \ldots, \beta_i, \alpha_{i+1}, \ldots, \alpha_n\})$ for $i = 1..n$. Due to $\alpha_i \equiv \beta_i$ and $\beta_i \nvdash \bot$, it is the case that $\{\alpha_i\} \nvdash \bot$ and $\{\alpha_i\} \equiv \{\alpha_i, \beta_i\}$. So, (Exchange) can be applied to $K \cup \{\beta_1, \ldots, \beta_{i-1}, \alpha_{i+1}, \ldots, \alpha_n\}$ for $K' = \{\alpha_i\}$ and $K'' = \{\alpha_i, \beta_i\}$. As a consequence, $I(K \cup \{\beta_1, \ldots, \beta_{i-1}, \alpha_i, \ldots, \alpha_n\})$ is then equal to $I(((K \cup \{\beta_1, \ldots, \beta_{i-1}, \alpha_{i+1}, \ldots, \alpha_n\}) \setminus \{\alpha_i\}) \cup \{\alpha_i, \beta_i\})$ and the latter is $I(K \cup \{\beta_1, \ldots, \beta_i, \alpha_{i+1}, \ldots, \alpha_n\})$. ■

That (Exchange) entails (Swap) is natural. More surprisingly, (Exchange) also entails (Tautology Independence) as the next result shows.

Proposition 8 *(Exchange) entails (Tautology Independence).*

Proof The non-trivial case is $\alpha \notin K$. Apply (Exchange′) for $K' = \{\alpha\}$ and $K'' = \emptyset$ so that $I(K \cup \{\alpha\}) = I((K \setminus \{\alpha\}) \cup \emptyset)$ ensues. I.e., $I(K \cup \{\alpha\}) = I(K)$. ■

5.2 The value of an adjunction postulate

In keeping with the meaning of the conjunction connective in classical logic, consider a dedicated postulate in the form

- $I(K \cup \{\alpha, \beta\}) = I(K \cup \{\alpha \wedge \beta\})$ (Adjunction Invariancy)

Proposition 9 *(Adjunction Invariancy) entails*

- $I(K \cup \{\alpha, \beta\}) = I((K \setminus \{\alpha, \beta\}) \cup \{\alpha \wedge \beta\})$ (Disjoint Adjunction Invariancy)
- $I(K) = I(\{\bigwedge K\})$ (Full Adjunction Invariancy) *where $\bigwedge K$ denotes $\alpha_1 \wedge \ldots \wedge \alpha_n$ for any enumeration $\alpha_1, \ldots, \alpha_n$ of K.*

Proof Let $K = \{\alpha_1, \ldots, \alpha_n\}$. Apply iteratively (Adjunction Invariancy) as $I(\{\alpha_1 \wedge \ldots \wedge \alpha_{i-1}, \alpha_i, \ldots, \alpha_n\}) = I(\{\alpha_1 \wedge \ldots \wedge \alpha_i, \alpha_{i+1}, \ldots, \alpha_n\})$ for $i = 2..n$. ■

Proposition 10 *Assuming $I(\{\alpha \wedge (\beta \wedge \gamma)\}) = I(\{(\alpha \wedge \beta) \wedge \gamma\})$ and $I(\{\alpha \wedge \beta\}) = I(\{\beta \wedge \alpha\})$, (Disjoint Adjunction Invariancy) and (Full Adjunction Invariancy) are equivalent.*

Proof Assume (Full Adjunction Invariancy). $K \cup \{\alpha, \beta\} = (K \setminus \{\alpha, \beta\}) \cup \{\alpha, \beta\}$ yields $I(K \cup \{\alpha, \beta\}) = I((K \setminus \{\alpha, \beta\}) \cup \{\alpha, \beta\})$. By (Full Adjunction Invariancy), $I((K \setminus \{\alpha, \beta\}) \cup \{\alpha, \beta\}) = I(\{\bigwedge((K \setminus \{\alpha, \beta\}) \cup \{\alpha, \beta\})\})$ and the latter can be written $I(\{\gamma_1 \wedge \ldots \wedge \gamma_n \wedge \alpha \wedge \beta\})$ for some enumeration $\gamma_1, \ldots, \gamma_n$ of $K \setminus \{\alpha, \beta\}$. I.e., $I(K \cup \{\alpha, \beta\}) = I(\{\gamma_1 \wedge \ldots \wedge \gamma_n \wedge \alpha \wedge \beta\})$. By (Full Adjunction Invariancy), $I((K \setminus \{\alpha, \beta\}) \cup \{\alpha \wedge \beta\}) = I(\{\bigwedge((K \setminus \{\alpha, \beta\}) \cup \{\alpha \wedge \beta\})\})$ that can be written $I(\{\gamma_1 \wedge \ldots \wedge \gamma_n \wedge \alpha \wedge \beta\})$ for the same enumeration $\gamma_1, \ldots, \gamma_n$ of $K \setminus \{\alpha, \beta\}$. So, $I(K \cup \{\alpha, \beta\}) = I((K \setminus \{\alpha, \beta\}) \cup \{\alpha \wedge \beta\})$. As to the converse, it is trivial to use (Disjoint Adjunction Invariancy) iteratively to get (Full Adjunction Invariancy). ■

A counter-example to the purported equivalence of (Adjunction Invariancy) and (Full Adjunction Invariancy) is as follows. Let $K = \{a, b, \neg b \wedge \neg a\}$. Obviously, $I(K \cup \{a, b\}) = I(K)$ since $\{a, b\} \subseteq K$. (Full Adjunction Invariancy) gives $I(K) = I(\{\bigwedge_{\gamma \in K} \gamma\})$ i.e. $I(K \cup \{a, b\}) = I(\{\bigwedge_{\gamma \in K} \gamma\}) = I(\{a \wedge b \wedge \neg b \wedge \neg a\})$. A different case of applying (Full Adjunction Invariancy) gives $I(K \cup \{a \wedge b\}) = I(\{\bigwedge_{\gamma \in K \cup \{a \wedge b\}} \gamma\}) = I(\{a \wedge b \wedge \neg b \wedge \neg a \wedge a \wedge b\})$. However, HK postulates do not provide grounds to infer $I(\{a \wedge b \wedge \neg b \wedge \neg a\}) = I(\{a \wedge b \wedge \neg b \wedge \neg a \wedge a \wedge b\})$ hence (Adjunction Invariancy) may fail here.

(Adjunction Invariancy) offers a natural equivalence between (Monotony) and the principle which expresses that adding a conjunct cannot make the amount of inconsistency to decrease :

Proposition 11 *Assuming (Consistency Null), (Adjunction Invariancy) implies that (Monotony) is equivalent with*

- $I(K \cup \{\alpha \wedge \beta\}) \geq I(K \cup \{\alpha\})$ (Conjunction Dominance)

Proof Assume (Monotony), an instance of which is $I(K \cup \{\alpha\}) \leq I(K \cup \{\alpha, \beta\})$. According to (Adjunction

Invariancy), $I(K \cup \{\alpha, \beta\}) = I(K \cup \{\alpha \wedge \beta\})$. Hence, $I(K \cup \{\alpha\}) \leq I(K \cup \{\alpha \wedge \beta\})$. That is, (Conjunction Dominance) holds.
Assume (Conjunction Dominance). First, consider $K \neq \emptyset$. Let $\alpha \in K$. Due to (Conjunction Dominance), $I(K \cup \{\alpha\}) \leq I(K \cup \{\alpha \wedge \beta\}$. (Adjunction Invariancy) gives $I(K \cup \{\alpha, \beta\}) = I(K \cup \{\alpha \wedge \beta\})$. Hence, $I(K \cup \{\alpha\}) \leq I(K \cup \{\alpha, \beta\})$. I.e., $I(K) \leq I(K \cup \{\beta\})$ since $\alpha \in K$. For $K' \in \mathcal{K}_{\mathcal{L}}$, it is enough to iterate this finitely many times (one for every β in $K' \setminus K$) in order to obtain $I(K) \leq I(K \cup K')$. Now, consider $K = \emptyset$. By (Consistency Null), $I(K) = 0$ hence $I(K) \leq I(K \cup K')$. ■

(Free Formula Independence) yields (Tautology Independence) by Proposition 2 although a more general principle (e.g., ($\top$-conjunct Independence) or the like) ensuring that I be independent of tautologies is to be expected. The next result shows that (Adjunction Invariancy) is the way to get both postulates at once.

Proposition 12 *Assuming (Consistency Null), (Adjunction Invariancy) implies that (Tautology Independence) and ($\top$-conjunct Independence) are equivalent.*

Proof For $\alpha \equiv \top$, (Adjunction Invariancy) and (Tautology Independence) give $I(K \cup \{\alpha \wedge \beta\}) = I(K \cup \{\alpha, \beta\}) = I(K \cup \{\beta\})$. As to the converse, let $\beta \in K$. Therefore, $I(K) = I(K \cup \{\beta\}) = I(K \cup \{\alpha \wedge \beta\}) = I(K \cup \{\alpha, \beta\}) = I(K \cup \{\alpha\})$. The case $K = \emptyset$ is settled by means of (Consistency Null). ■

Lastly, (Adjunction Invariancy) provides for free various principles related to (idempotence, commutativity, and associativity of) conjunction, as follows.

Proposition 13 *(Adjunction Invariancy) entails*

- $I(K \cup \{\alpha \wedge \alpha\}) = I(K \cup \{\alpha\})$
- $I(K \cup \{\alpha \wedge \beta\}) = I(K \cup \{\beta \wedge \alpha\})$
- $I(K \cup \{\alpha \wedge (\beta \wedge \gamma)\}) = I(K \cup \{(\alpha \wedge \beta) \wedge \gamma\})$

Proof (i) $I(K \cup \{\alpha \wedge \alpha\}) = I(K \cup \{\alpha, \alpha\}) = I(K \cup \{\alpha\})$. (ii) $I(K \cup \{\alpha \wedge \beta\}) = I(K \cup \{\alpha, \beta\}) = I(K \cup \{\beta, \alpha\}) = I(K \cup \{\beta \wedge \alpha\})$. (iii) $I(K \cup \{\alpha \wedge (\beta \wedge \gamma)\}) = I(K \cup \{\alpha, \beta \wedge \gamma\}) = I(K \cup \{\alpha, \beta, \gamma\}) = I(K \cup \{\alpha \wedge \beta, \gamma\}) = I(K \cup \{(\alpha \wedge \beta) \wedge \gamma\})$. ■

(Adjunction Invariancy) and (Exchange) are two principles devoted to ensuring that replacing a subset of the knowledge base with an equivalent subset does not change the value given by the inconsistency measure. The contexts that these two principles require for the replacement to be safe differ :

1. For $K' \nvdash \bot$, (Exchange) is more general than (Adjunction Invariancy) since (Exchange) guarantees $I(K \cup K') = I(K \cup K'')$ for every $K'' \equiv K'$ but (Adjunction Invariancy) ensures it only for $K'' = \{\bigwedge K'_i \mid \mathfrak{K} = \{K'_1, .., K'_n\}\}$ where $\mathfrak{K}$ ranges over the partitions of K'.
2. For $\alpha \vdash \bot$, (Adjunction Invariancy) is more general than (Exchange) because (Adjunction Invariancy) guarantees $I(K \cup \{\alpha, \beta\}) = I(K \cup \{\alpha \wedge \beta\})$ but (Exchange) does not guarantee it.

6 Revisiting HK Postulates

6.1 Sticking with (Consistency Null) and (Monotony)

(Consistency Null) or a like postulate is indispensable because there seems to be no way to have a sensible inconsistency measure that would not be able to always discriminate between consistency and inconsistency.

(Monotony) is to be kept since contradictions in classical logic (and basically all logics) are monotone [1] wrt information : i.e., extra information cannot make a contradiction to vanish.

However, we will not retain (Monotony) as an explicit postulate, because it ensues from the postulate to be introduced in Section 6.4.

6.2 Intended postulates

In addition, both (Tautology Independence) and ($\top$-conjunct Independence) are due postulates. Even more generally, it would make no sense, when considering how inconsistent a theory is, to take into account any inessential difference in which a formula is written (for example, $\alpha \vee \beta$ instead of $\beta \vee \alpha$). Define α' to be a *prenormal form* of α if α' results from α by applying (possibly repeatedly) one or more of these principles : commutativity, associativity and distribution for $\wedge$ and $\vee$, De Morgan laws, double negation equivalence. Hence the next[4] postulate :

- If β is a prenormal form of α then $I(K \cup \{\alpha\}) = I(K \cup \{\beta\})$ (Rewriting)

As (Monotony) essentially means that extra information cannot make amount of inconsistency to decrease, the same idea must apply to conjunction because $\alpha \wedge \beta$ does involve more information than α. Thus, another due postulate is :

[4] In sharp contrast to (Irrelevance of Syntax) that allows for destructive transformation from α to β when both are inconsistent, (Rewriting) takes care of inhibiting purely deductive transformations (the most important one is presumably from $\alpha \wedge \bot$ to $\bot$).

- $I(K \cup \{\alpha \wedge \beta\}) \geq I(K \cup \{\alpha\})$ (Conjunction Dominance)

Indeed, it does not matter whether α or β or both be inconsistent : It definitely cannot be rational to hold that there is a case (even a single one) where extending K with a conjunction would result in *less* inconsistency than extending K with one of the conjuncts.

6.3 Taking care of disjunction

It is a delicate matter to assess how inconsistent a disjunction is, but bounds can be set. Indeed, a disjunction expresses two alternative possibilities, so that accrual across these would make little sense. That is, amount of inconsistency in $\alpha \vee \beta$ cannot exceed amount of inconsistency in either α or β, depending on which one involves a higher amount of inconsistency. Hence the next postulate.

- $I(K \cup \{\alpha \vee \beta\}) \leq \max(I(K \cup \{\alpha\}), I(K \cup \{\beta\}))$ (Disjunct Maximality)

Proposition 14 *Assuming $I(K \cup \{\alpha \vee \beta\}) = I(K \cup \{\beta \vee \alpha\})$, it is the case that (Disjunct Maximality) is equivalent with each of*

- *if $I(K \cup \{\alpha\}) \geq I(K \cup \{\beta\})$ then $I(K \cup \{\alpha\}) \geq I(K \cup \{\alpha \vee \beta\})$*
- *$I(K \cup \{\alpha \vee \beta\}) \leq I(K \cup \{\alpha\})$ or $I(K \cup \{\alpha \vee \beta\}) \leq I(K \cup \{\beta\})$*

Proof Let us prove that (Disjunct Maximality) entails the first item. Assume $I(K \cup \{\alpha\}) \geq I(K \cup \{\beta\})$. I.e., $I(K \cup \{\alpha\}) = \max(I(K \cup \{\alpha\}), I(K \cup \{\beta\}))$. Using (Disjunct Maximality), $I(K \cup \{\alpha \vee \beta\}) \leq \max(I(K \cup \{\alpha\}), I(K \cup \{\beta\}))$, i.e. $I(K \cup \{\alpha\})) \geq I(K \cup \{\alpha \vee \beta\})$. As to the converse direction, assume that if $I(K \cup \{\alpha\}) \geq I(K \cup \{\beta\})$ then $I(K \cup \{\alpha\}) \geq I(K \cup \{\alpha \vee \beta\})$. Consider the case $\max(I(K \cup \{\alpha\}), I(K \cup \{\beta\})) = I(K \cup \{\alpha\})$. Hence, $I(K \cup \{\alpha\}) \geq I(K \cup \{\beta\})$. According to the assumption, it follows that $I(K \cup \{\alpha\}) \geq I(K \cup \{\alpha \vee \beta\})$. That is, $\max(I(K \cup \{\alpha\}), I(K \cup \{\beta\})) \geq I(K \cup \{\alpha \vee \beta\})$. Similarly, the case $\max(I(K \cup \{\alpha\}), I(K \cup \{\beta\})) = I(K \cup \{\beta\})$ gives $I(K \cup \{\beta\}) \geq I(K \cup \{\beta \vee \alpha\})$. Then, $I(K \cup \{\beta\}) \geq I(K \cup \{\alpha \vee \beta\})$ in view of the hypothesis in the statement of Proposition 14. That is, $\max(I(K \cup \{\alpha\}), I(K \cup \{\beta\})) \geq I(K \cup \{\alpha \vee \beta\})$. Combining both cases, (Disjunct Maximality) holds.
The equivalence of (Disjunct Maximality) with the last item is due to the fact that the codomain of I is totally ordered. ■

Although it is quite unclear how to weigh inconsistencies out of a disjunction, they must weigh somewhat less than out of both disjuncts (whether tied together by a conjunction or not), which is the reason for holding

- $I(K \cup \{\alpha \wedge \beta\}) \geq I(K \cup \{\alpha \vee \beta\})$ ($\wedge$-over-$\vee$ Dominance)

and its conjunction-free counterpart

- $I(K \cup \{\alpha, \beta\}) \geq I(K \cup \{\alpha \vee \beta\})$

Proposition 15 *Assuming $I(K \cup \{\alpha \wedge \beta\}) = I(K \cup \{\beta \wedge \alpha\})$, (Conjunction Dominance) and (Disjunct Maximality) entail ($\wedge$-over-$\vee$ Dominance).*

Proof Given $I(K \cup \{\alpha \wedge \beta\}) = I(K \cup \{\beta \wedge \alpha\})$, (Conjunction Dominance) gives $I(K \cup \{\alpha \wedge \beta\}) \geq I(K \cup \{\alpha\})$ and $I(K \cup \{\alpha \wedge \beta\}) \geq I(K \cup \{\beta\})$. Therefore, $\max(I(K \cup \{\alpha\}), I(K \cup \{\beta\})) \leq I(K \cup \{\alpha \wedge \beta\})$. In view of (Disjunct Maximality), $I(K \cup \{\alpha \vee \beta\}) \leq \max(I(K \cup \{\alpha\}), I(K \cup \{\beta\}))$, and it accordingly follows that $I(K \cup \{\alpha \vee \beta\}) \leq I(K \cup \{\alpha \wedge \beta\})$ holds. ■

Proposition 16 *(Monotony) and (Disjunct Maximality) entail*

- $I(K \cup \{\alpha, \beta\}) \geq I(K \cup \{\alpha \vee \beta\})$

Proof $I(K \cup \{\alpha\}) \leq I(K \cup \{\alpha, \beta\})$ and $I(K \cup \{\beta\}) \leq I(K \cup \{\alpha, \beta\})$ according to (Monotony). Consequently, $\max(I(K \cup \{\alpha\}), I(K \cup \{\beta\})) \leq I(K \cup \{\alpha, \beta\})$. Due to (Disjunct Maximality), $I(K \cup \{\alpha \vee \beta\}) \leq \max(I(K \cup \{\alpha\}), I(K \cup \{\beta\}))$. Therefore, $I(K \cup \{\alpha, \beta\}) \geq I(K \cup \{\alpha \vee \beta\})$. ■

For the record, another plausible postulate related to disjunction is

- $I(K \cup \{\alpha \vee \beta\}) \geq \min(I(K \cup \{\alpha\}), I(K \cup \{\beta\}))$ (Disjunct Minimality)

Similarly to Proposition 14, the following ensues :

Proposition 17 *Assuming $I(K \cup \{\alpha \vee \beta\}) = I(K \cup \{\beta \vee \alpha\})$, (Disjunct Minimality) is equivalent to each of*

- *if $I(K \cup \{\beta\}) \geq I(K \cup \{\alpha\})$ then $I(K \cup \{\alpha \vee \beta\}) \geq I(K \cup \{\alpha\})$*
- *$I(K \cup \{\alpha \vee \beta\}) \geq I(K \cup \{\alpha\})$ or $I(K \cup \{\alpha \vee \beta\}) \geq I(K \cup \{\beta\})$*

6.4 A schematic postulate

The next postulate is to be presented in two steps.

1. (Monotony) expresses that adding information cannot result in a decrease of the amount of inconsistency in the knowledge base. Considering a notion of primitive conflicts that underlies amount of inconsistency, (Monotony) is a special case of a postulate stating that amount of inconsistency is monotone with respect to the set of primitive conflicts $\mathcal{C}(K)$ of the knowledge base K : If $\mathcal{C}(K) \subseteq \mathcal{C}(K')$ then $I(K) \leq I(K')$.

Clearly, I is to admit different postulates depending on what features are required for primitive conflicts (see Table 1).

2. Keep in mind that an inconsistency measure refers to logical content of the knowledge base, *not* other aspects whether subject matter of contradiction, source of information,... This is because an inconsistency measure is only concerned with *quantity*, i.e. amount of inconsistency (of course, it is possible for example that a contradiction be more worrying than another -and so, making more pressing *to act* [4] about it- but this has nothing to do with amount of inconsistency). Now, what characterizes logical content is uniform substitutivity. Hence a postulate called (Substitutivity Dominance) stating that renaming cannot make the amount of inconsistency to decrease : If $\sigma K = K'$ for some substitution σ then $I(K) \leq I(K')$.

Combining these two ideas, we obtain the following postulate

- If $\mathcal{C}(\sigma K) \subseteq \mathcal{C}(K')$ for some substitution σ then $I(K) \leq I(K')$

(Subsumption Orientation)

Fact 1 *Every postulate of the form*

- *$I(X) \leq I(Y)$ for all $X \in \mathcal{K}_{\mathcal{L}}$ and $Y \in \mathcal{K}_{\mathcal{L}}$ such that condition $C_{X,Y}$ holds*

or of the form

- *$I(X) = I(Y)$ for all $X \in \mathcal{K}_{\mathcal{L}}$ and $Y \in \mathcal{K}_{\mathcal{L}}$ such that condition $C_{X,Y}$ holds*

is derived from (Subsumption Orientation) and from any property of $\mathcal{C}$ ensuring that condition C holds.

Individual properties of $\mathcal{C}$ ensuring condition C for a number of postulates, including all those previously mentioned in the paper, can be found in Table 1. Please notice that the fact that (Instance Low) and (Monotony) share the same requirement over $\mathcal{C}$ does *not* mean that the *postulate* (Monotony) entails the *postulate* (Instance Low).

Proposition 18 *Assuming $\mathcal{C}(K) \subseteq \mathcal{C}(K \cup K')$ for all $K \in \mathcal{K}_{\mathcal{L}}$ and $K' \in \mathcal{K}_{\mathcal{L}}$, (Subsumption Orientation) yields the following derived postulate :*

- *If $\sigma K \subseteq K'$ for some substitution σ then $I(K) \leq I(K')$* *(Instance Low)*

Proof Given the property $\mathcal{C}(X) \subseteq \mathcal{C}(X \cup Y)$ for all $X \in \mathcal{K}_{\mathcal{L}}$ and for all $Y \in \mathcal{K}_{\mathcal{L}}$, we must prove that $I(K) \leq I(K')$ holds whenever there exists some substitution σ such that $\sigma K \subseteq K'$. Assume $\sigma K \subseteq K'$. Using the property just mentioned, $\mathcal{C}(\sigma K) \leq \mathcal{C}(K')$ ensues (since $K' \setminus \sigma K = \{\alpha_1, \ldots, \alpha_n\}$ is finite due to $K' \in \mathcal{K}_{\mathcal{L}}$). Applying (Subsumption Orientation), $I(K) \leq I(K')$. ■

Proposition 18 is merely a special case of Fact 1 but it is stated explicitly because (Monotony) is entailed by (Instance Low).

6.5 A new system of postulates (basic version and strong version)

All the above actually suggests a new system of postulates, which consists simply of (Consistency Null) and (Subsumption Orientation). The system is parameterized by the properties imposed upon $\mathcal{C}$ in the latter. In the range thus induced by $\mathcal{C}$, a basic system emerges, which amounts to the list in Table 2. At the other end of the range is the strong system in Table 3. Except for (Dominance) and (Free Formula Independence), it captures all the postulates listed in Table 1. It is trivial to check that both systems are coherent.

7 HK Postulates identified as (Subsumption Orientation)

Time has come to make sense[5] of the HK choice of (Free Formula Independence) together with (Monotony), by means of Theorem 1 and Theorem 2.

Theorem 1 *Let $\mathcal{C}$ be such that for every $K \in \mathcal{K}_{\mathcal{L}}$ and for every $X \subseteq \mathcal{L}$ which is minimal inconsistent, $X \in \mathcal{C}(K)$ iff $X \subseteq K$. If I satisfies both (Monotony) and (Free Formula Independence) then I satisfies (Subsumption Orientation) restricted to its non-substitution part, namely*

$$\text{if } \mathcal{C}(K) \subseteq \mathcal{C}(K') \text{ then } I(K) \leq I(K').$$

Proof Let $\mathcal{C}(K) \subseteq \mathcal{C}(K')$. Should K be a subset of K', (Monotony) yields $I(K) \leq I(K')$ as desired. So, let us turn to $K \not\subseteq K'$. Consider $\varphi \in K \setminus K'$. If φ were not free for K, there would exist a minimal inconsistent subset X of K such that $\varphi \in X$. Clearly, $X \not\subseteq K'$. The constraint imposed on $\mathcal{C}$ in the statement of the theorem would then yield both $X \in \mathcal{C}(K)$ and $X \notin \mathcal{C}(K')$, contradicting the assumption $\mathcal{C}(K) \subseteq \mathcal{C}(K')$. Hence, φ is free for K. In view of (Free Formula Independence), $I(K) = I(K \setminus \{\varphi\})$. The same reasoning applied to all the (finitely many) formulas in $K \setminus K'$ gives $I(K) = I(K \cap K')$. However, $K \cap K'$ is a subset of K' so that using (Monotony) yields $I(K \cap K') \leq I(K')$ hence $I(K) \leq I(K')$. ■

Define $\Xi = \{X \in \mathcal{K}_{\mathcal{L}} \mid \forall X' \subseteq X, X' \vdash \bot \Leftrightarrow X = X'\}$. Then, $\mathcal{C}$ is said to be *governed by minimal inconsistency* iff $\mathcal{C}$ satisfies the following property

$$\text{if } \mathcal{C}(K) \cap \Xi \subseteq \mathcal{C}(K') \cap \Xi \text{ then } \mathcal{C}(K) \subseteq \mathcal{C}(K').$$

[5] Although still not defending the choice of (Free Formula Independence).

Specific property for $\mathcal{C}$	*Specific postulate entailed by (Subsumption Orientation)*
No property needed	(Variant Equality)
No property needed	(Substitutivity Dominance)
$\mathcal{C}(K \cup \{\alpha\}) = \mathcal{C}(K)$ for $\alpha \equiv \top$	(Tautology Independence)
$\mathcal{C}(K \cup \{\alpha \wedge \beta\}) = \mathcal{C}(K \cup \{\beta\})$ for $\alpha \equiv \top$	($\top$-conjunct Independence)
$\mathcal{C}(K \cup \{\alpha\}) = \mathcal{C}(K \cup \{\alpha'\})$ for α' prenormal form of α	(Rewriting)
$\mathcal{C}(K) \subseteq \mathcal{C}(K \cup \{\alpha\})$	(Instance Low)
$\mathcal{C}(K) \subseteq \mathcal{C}(K \cup \{\alpha\})$	(Monotony)
$\mathcal{C}(K \cup \{\alpha \vee \beta\}) \subseteq \mathcal{C}(K \cup \{\alpha \wedge \beta\})$	($\wedge$-over-$\vee$ Dominance)
$\mathcal{C}(K \cup \{\alpha\}) \subseteq \mathcal{C}(K \cup \{\alpha \wedge \beta\})$	(Conjunction Dominance)
$\mathcal{C}(K \cup \{\alpha, \beta\}) = \mathcal{C}(K \cup \{\alpha \wedge \beta\})$	(Adjunction Invariancy)
$\mathcal{C}(K \cup \{\alpha \vee \beta\}) \subseteq \mathcal{C}(K \cup \{\alpha\})$ or $\mathcal{C}(K \cup \{\beta\})$	(Disjunct Maximality)
$\mathcal{C}(K \cup \{\alpha \vee \beta\}) \supseteq \mathcal{C}(K \cup \{\alpha\})$ or $\mathcal{C}(K \cup \{\beta\})$	(Disjunct Minimality)
$\mathcal{C}(K \cup K') = \mathcal{C}(K \cup K'')$ for $K'' \equiv K' \nvdash \bot$	(Exchange)
$\mathcal{C}(K \cup \{\alpha_1, ..., \alpha_n\}) = \mathcal{C}(K \cup \{\beta_1, .., \beta_n\})$ if $\alpha_i \equiv \beta_i \nvdash \bot$	(Swap)
$\mathcal{C}(K \cup \{\beta\}) \subseteq \mathcal{C}(K \cup \{\alpha\})$ for $\alpha \vdash \beta$ and $\alpha \nvdash \bot$	(Dominance)
$\mathcal{C}(K \cup \{\alpha\}) = \mathcal{C}(K)$ for α free for K	(Free Formula Independence)

TAB. 1 – Conditions for postulates derived from (Subsumption Orientation).

(Variant Equality) is named after the notion of a variant [2] :

- if $\sigma K = K'$ and $\sigma' K' = K$ for some substitutions σ and σ' then $I(K) = I(K')$ (Variant Equality)

Basic System

$I(K) = 0$ iff $K \nvdash \bot$	(Consistency Null)
If α' is a prenormal form of α then $I(K \cup \{\alpha\}) = I(K \cup \{\alpha'\})$	(Rewriting)
If $\sigma K \subseteq K'$ for some substitution σ then $I(K) \leq I(K')$	(Instance Low)
$I(K \cup \{\alpha \vee \beta\}) \leq \max(I(K \cup \{\alpha\}), I(K \cup \{\beta\}))$	(Disjunct Maximality)
If $\alpha \equiv \top$ then $I(K) = I(K \cup \{\alpha\})$	(Tautology Independence)
If $\alpha \equiv \top$ then $I(K \cup \{\alpha \wedge \beta\}) = I(K \cup \{\beta\})$	($\top$-conjunct Independence)
$I(K \cup \{\alpha\}) \leq I(K \cup \{\alpha \wedge \beta\})$	(Conjunction Dominance)

TAB. 2 – Basic system

Strong System

$I(K) = 0$ iff $K \nvdash \bot$	(Consistency Null)
If α' is a prenormal form of α then $I(K \cup \{\alpha\}) = I(K \cup \{\alpha'\})$	(Rewriting)
If $\sigma K \subseteq K'$ for some substitution σ then $I(K) \leq I(K')$	(Instance Low)
$I(K \cup \{\alpha \vee \beta\}) \leq \max(I(K \cup \{\alpha\}), I(K \cup \{\beta\}))$	(Disjunct Maximality)
$I(K \cup \{\alpha \vee \beta\}) \geq \min(I(K \cup \{\alpha\}), I(K \cup \{\beta\}))$	(Disjunct Minimality)
If $K'' \equiv K' \nvdash \bot$ then $I(K) = I(K \cup K'')$	(Exchange)
$I(K \cup \{\alpha, \beta\}) = I(K \cup \{\alpha \wedge \beta\})$	(Adjunction Invariancy)

TAB. 3 – Strong system

Please note that $\mathcal{C}$ being governed by minimal inconsistency does not mean that $\mathcal{C}(K)$ is determined by the set of minimal inconsistent subsets of K. Intuitively, it only means that those Z in $\mathcal{C}(K)$ which are not minimal inconsistent cannot override set-inclusion induced by minimal inconsistent subsets —i.e., no such Z can, individually or collectively, turn $\mathcal{C}(K) \cap \Xi \subseteq \mathcal{C}(K') \cap \Xi$ into $\mathcal{C}(K) \not\subseteq \mathcal{C}(K')$.

Theorem 2 *Let $\mathcal{C}$ be governed by minimal inconsistency and be such that for all $K \in \mathcal{K}_{\mathcal{L}}$ and all $X \subseteq \mathcal{L}$ which is minimal inconsistent, $X \in \mathcal{C}(K)$ iff $X \subseteq K$. I satisfies (Monotony) and (Free Formula Independence) whenever I satisfies (Subsumption Orientation) restricted to its non-substitution part, namely*

$$\text{if } \mathcal{C}(K) \subseteq \mathcal{C}(K') \text{ then } I(K) \leq I(K').$$

Proof Trivially, if $X \subseteq K$ then $X \subseteq K \cup \{\alpha\}$. By the constraint imposed on $\mathcal{C}$ in the statement of the theorem, it follows that if $X \in \mathcal{C}(K)$ then $X \in \mathcal{C}(K \cup \{\alpha\})$. Since $\mathcal{C}$ is governed by minimal inconsistency, $\mathcal{C}(K) \subseteq \mathcal{C}(K \cup \{\alpha\})$ ensues and (Subsumption Orientation) yields (Monotony). Let α be a free formula for K. By definition, α is in no minimal inconsistent subset of $K \cup \{\alpha\}$. So, $X \subseteq K$ iff $X \subseteq K \cup \{\alpha\}$ for all minimal inconsistent X. By the constraint imposed on $\mathcal{C}$ in the statement of the theorem, $X \in \mathcal{C}(K)$ iff $X \in \mathcal{C}(K \cup \{\alpha\})$ ensues for all minimal inconsistent X. In symbols, $\mathcal{C}(K) \cap \Xi = \mathcal{C}(K \cup \{\alpha\}) \cap \Xi$. Since $\mathcal{C}$ is governed by minimal inconsistency, it follows that $\mathcal{C}(K) = \mathcal{C}(K \cup \{\alpha\})$. Thus, (Free Formula Independence) holds, due to (Subsumption Orientation). ■

Therefore, Theorem 1 and Theorem 2 mean that, *if substitutivity is left aside*, (Subsumption Orientation) is equivalent with (Free Formula Independence) and (Monotony) when primitive conflicts are essentially minimal inconsistent subsets. So, these postulates form a natural pair *if it is assumed that* minimal inconsistent subsets must be the basis for inconsistency measuring.

8 Conclusion

We have proposed a new system of postulates for inconsistency measures, i.e.

- $I(K) = 0$ *iff* K *is consistent* (Consistency Null)
- *If* $\mathcal{C}(\sigma K) \subseteq \mathcal{C}(K')$ *for some substitution* σ *then* $I(K) \leq I(K')$ (Subsumption Orientation)

parameterized by the requirements imposed on $\mathcal{C}$.

Even in its strong version, the new system omits both (Dominance) and (Free Formula Independence), which we have argued against. We have investigated various postulates, absent from the HK set, giving grounds to include them in the new system. Lastly, we have shown that (Subsumption Orientation) not only accounts for the other postulates but also provides a justification for (Free Formula Independence) together with (Monotony), through focussing on minimal inconsistent subsets.

We do not hold that the new system, in basic or strong version, captures all desirable cases, we more modestly claim for improving over the original HK set. In particular, we believe that the HK postulates suffer from over-commitment to minimal inconsistent subsets. Crucially, such a comment applies to *postulates* (because they would exclude all approaches that are not based upon minimal inconsistent subsets) but it does not apply to *measures* themselves : There are excellent reasons to develop a specific measure [13] [15] [19] ... based on minimal inconsistent sets (in contrast, a set of postulates must take care of generality).

As to future work, we must mention taking seriously belief bases as multisets. Perhaps the most insightful postulate in this respect is (Adjunction Invariancy) since there surely is some rationality in holding that $\{a \wedge b \wedge \neg a \wedge \neg b \wedge a \wedge b \wedge \neg a \wedge \neg b\}$ is more inconsistent than $\{a \wedge b \wedge \neg a \wedge \neg b\}$.

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